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Technical Notes

Improved Exponential Integral Approximation for Tangent-Slab Radiation Transport

Christopher O. Johnston*
NASA Langley Research Center, Hampton, Virginia 23681

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Nomenclature

a = researcher-dependent constant for the E_3 approximation
E_3 = third-order exponential integral
h = Planck's constant, 6.6256×10^{-27} erg-s
j_ν = frequency-dependent emission coefficient, erg/cm^3-sr
q_ν^- = frequency-dependent wall-directed radiative flux, W-s/cm^2
q^- = frequency-integrated wall-directed radiative flux, or just the radiative flux for constant-property-layer cases, W/cm^2
s = integration variable, cm
T = temperature, K
x = optical depth for a constant property layer, equal to $\kappa_\nu z$, dimensionless
z = distance along a line of sight, cm
z_s = shock location or edge of a constant property layer, cm
κ_ν = frequency-dependent absorption coefficient, cm^{-1}
ν = frequency, s^{-1}
τ = optical thickness, dimensionless
ω = integration variable

I. Introduction

THE tangent-slab approximation is often used to compute shock-layer radiation for reentry vehicles [1]. With this approximation, the radiative flux directed to the vehicle wall is a function of the properties along the line of sight normal to the body. Thus, the wall-directed radiative flux at point z along the line of sight is written as [2]

$$q_\nu^-(z) = 2\pi \int_{s=z}^{s=z_s} j_\nu(s) \left| \frac{\mathrm{d}E_3[\tau(s,z)]}{\mathrm{d}\tau} \right| \mathrm{d}s \tag{1}$$

where z_s is the outer limit of integration (such as the shock location), $\tau(s,z)$ is the optical thickness defined as

$$\tau(s,z) = \int_s^z \kappa_\nu \, \mathrm{d}z \tag{2}$$

and j_ν and κ_ν are the emission and absorption coefficients, respectively. The function E_3 in Eq. (1) represents the third-order exponential integral, defined as

$$E_3(x) = \int_1^\infty \omega^{-3} \exp(-\omega x) \, \mathrm{d}\omega \tag{3}$$

for which there is no exact analytic solution.

The numerical solution of Eqs. (1–3) is required at each spatial and spectral point in the shock layer, which for typical cases consists of more than 1×10^6 points. As a result of this large number of points, an efficient solution procedure is desired for these equations. A common approach [3,4] is to approximate E_3 with an exponential function of the following form:

$$E_3(x) = \tfrac{1}{2} e^{-ax} \tag{4}$$

where a is a constant. Values for a have been cited by numerous researchers: Hunt and Sibulkin (HS) [5], $a = 2.0$; Siegel and Howell (SH) [6], $a = 1.8$; Ozisik (OZ) [7], $a = 1.562$; and Modest (MO) [8], $a = 1.5$. Another approximation was presented by Murty (MU) [9] in a slightly different form:

$$E_3(x) = 0.30 e^{-1.1613x} + 0.222 e^{-2.941x} \tag{5}$$

The exponential form of these approximations allows for simplifications in the numerical evaluation of Eqs. (1–3). This is seen by substituting Eq. (4) into Eq. (1), resulting in

$$q_\nu^-(z) = \pi a e^{-a\tau(0,z)} \int_{s=z}^{s=z_s} j_\nu(s) e^{a\tau(0,s)} \, \mathrm{d}s \tag{6}$$

This form is advantageous because the integrand is not a function of z, which significantly simplifies its numerical evaluation. An analogous form of Eq. (6) may also be written using Eq. (5). Note that other proposed nonexponential E_3 approximations [10–12] do not allow for this simplification.

Figure 1 compares the exact values of E_3 with the HS [5], SH [6], and MU [9] approximations. A noticeable disagreement with the exact function is seen for the HS and SH approximations, while the MU approximation agrees relatively well. Although not shown in Fig. 1, similar disagreement is obtained using the OZ [7] and MO [8] approximations. The influence of this disagreement on shock-layer radiative flux predictions may be assessed by assuming the shock layer is a constant property layer. With this assumption, Eq. (1) may be integrated analytically to obtain the following:

$$q_\nu^-(z) = 2\pi \frac{j_\nu}{\kappa_\nu} [E_3(0) - E_3(x)] \tag{7}$$

where $x = \tau(0,z)$, which is equal to $\kappa_\nu z$ (and z is the thickness of the layer). By studying Eq. (7), the qualities of an E_3 approximation that will lead to an accurate prediction of shock-layer radiative flux may be obtained. For instance, observing that $E_3(0) = 0.5$ and $E_3(1.5) = 0.0567$, it is concluded that the accuracy of E_3 for $x < 1.5$ is more important than for $x > 1.5$. This is true because the contribution of $E_3(x)$ to the bracketed difference in Eq. (7) is small for $x > 1.5$. Next, note that for optically thin conditions ($x \ll 1$), Eq. (7) may be expanded in a Taylor series as

$$q_\nu^-(x \ll 1) = -2\pi \frac{j_\nu}{\kappa_\nu} x \frac{\mathrm{d}E_3(0)}{\mathrm{d}x} + \mathcal{O}(x^2) \tag{8}$$

This shows that the derivative of E_3 at $x = 0$ must be accurate for the E_3 approximation to accurately model optically thin conditions. Lastly, for optically thick conditions ($x \gg 1$), Eq. (7) reduces to

$$q_\nu^-(z) = 2\pi \frac{j_\nu}{\kappa_\nu} E_3(0) + \mathcal{O}(e^{-x}/x) \tag{9}$$

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*Aerospace Engineer. Member AIAA.

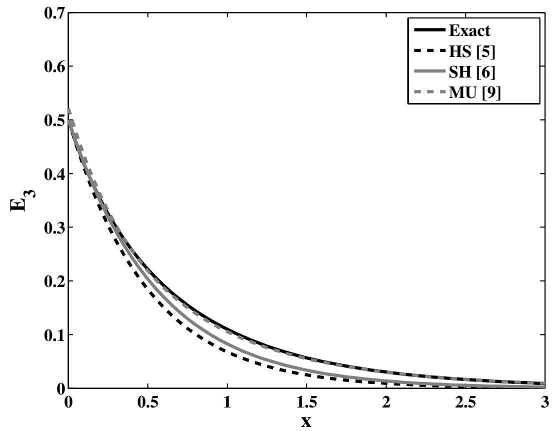

Fig. 1 Comparison of the exact values of E_3 with various approximations.

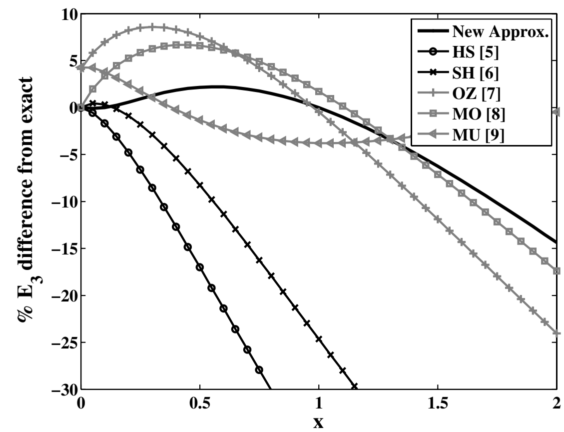

Fig. 2 Comparison of the percent differences between the exact and approximate E_3 functions.

which implies that the value of E_3 at $x = 0$ must be accurate for the approximation to accurately model optically thick conditions.

The previous paragraph suggests the following three criteria for developing an E_3 approximation that provides an accurate prediction of shock-layer radiative flux:

1) The accuracy of E_3 for $x < 1.5$ is more important than for larger x values.

2) The E_3 derivative at $x = 0$ must be accurately modeled.

3) The E_3 value at $x = 0$ must be accurately modeled.

Of the previous approximations mentioned in the discussion of Eqs. (4) and (5), all except the MU approximation satisfy criterion 2, whereas only the HS and MU approximations satisfy criterion 3. Regarding criterion 1, it will be shown in the next section that all mentioned approximations, except the MU approximation, have maximum errors greater than 5% for x values below 1.5. The subject of the remainder of this Note is to find an approximation that has an error of less than 5% for values of x below 1.5 while also satisfying criteria 2 and 3.

II. Improved E_3 Approximation

The improved E_3 approximation developed in this work uses an equation of the form of Eq. (5). The four coefficients of this equation are obtained by forcing the approximation to satisfy the following four equations: $E_3(0) = 1/2$, $\mathrm{d}E_3(0)/\mathrm{d}x = 1$, $E_3(0.1) = 0.4163$, and $E_3(1.0) = 0.1097$. The first two of these equations enforce criteria 2 and 3, as defined in the previous section, and the last two enforce criterion 1. The resulting approximation is written as

$$E_3(x) = 0.4661e^{-1.447x} + 0.0339e^{-9.614x} \tag{10}$$

Although not shown in Fig. 1, the results of Eq. (10) are nearly indistinguishable from the exact function for values of x below 1.0. The percent differences between the exact function and this new approximation, as well as those of [5–9], are shown in Fig. 2 as a function of x. It is seen that the new approximation has an error of less than 2.5% for values of x below 1.0, and an error of 6% at $x = 1.5$, which is only slightly larger than the desired 5%.

III. Application of the New E_3 Approximation

To show the impact of the new E_3 approximation on shock-layer radiation predictions, a set of constant property layers of equilibrium air at a range of temperatures (6000–14000 K), pressures (0.1–1.0 atm), and layer thicknesses (1–20 cm) are considered. These conditions are relevant to the shock layer of a lunar-return vehicle, which is typically dominated by equilibrium radiation at a similar pressure, temperature, and shock-layer thickness [13]. The exact radiative flux equation for a constant property layer was presented previously as Eq. (7). The required emission and absorption coefficients are computed using the HARA code of [14].

The absolute value of the percent difference between the radiative flux predicted by the exact E_3 function and the various E_3 approximations are shown in Fig. 3 for a range of temperatures. The pressure and layer thickness are set to 1 atm and 20 cm, respectively. It is seen that the new approximation is more accurate than the other approximations for the entire range of temperatures considered. A similar comparison is shown in Fig. 4 for a range of layer thicknesses (z), with the pressure and temperature set to 0.1 atm and 8000 K, respectively. Again, it is seen that the new approximation is more

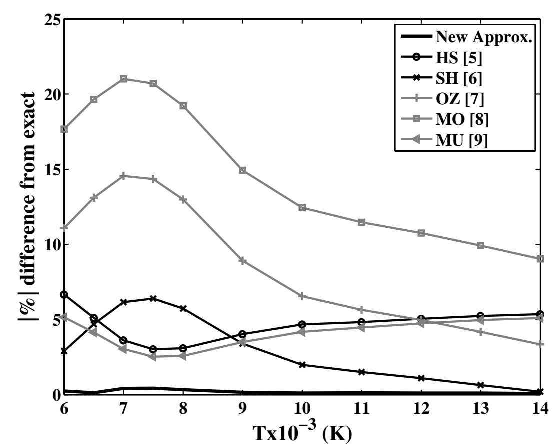

Fig. 3 Comparison of the percent differences in the radiative flux predicted by the various approximations for a 20 cm constant property slab of equilibrium air at 1.0 atm and a range of temperatures.

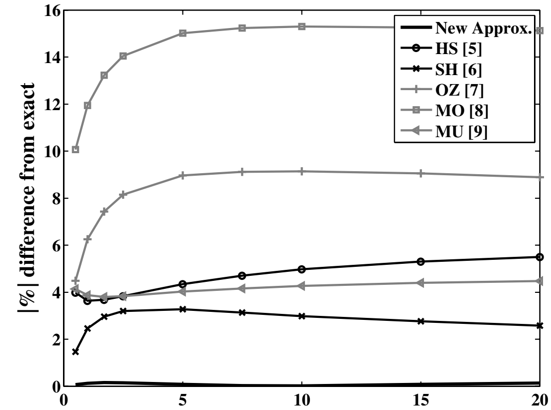

Fig. 4 Comparison of the percent differences in the radiative flux predicted by the various approximations for a 8000 K constant property slab of equilibrium air at 0.1 atm and a range of layer thicknesses.

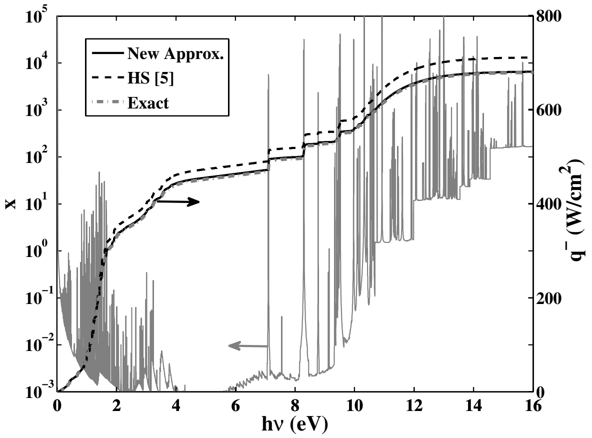

Fig. 5 Optical depth and radiative flux values resulting from a 20 cm constant property slab of equilibrium air at 1.0 atm and 10,000 K.

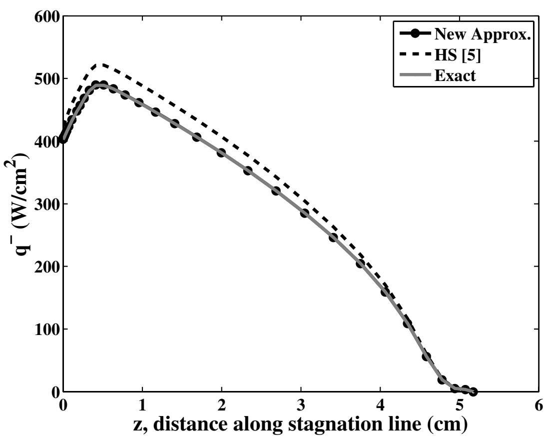

Fig. 6 Comparison of the wall-directed radiative flux along the stagnation line of a 1-m-radius sphere at 11 km/s.

accurate than the other approximations for the entire range of z values considered. In both Figs. 3 and 4, the percent error of the new approximation is less than 0.5%.

The optical depth x and radiative flux resulting from the HS and new approximations are compared with the exact function in Fig. 5 for a case with a temperature, pressure, and thickness of 10,000 K, 1.0 atm, and 20 cm, respectively. This figure shows that the new approximation and exact results are nearly indistinguishable, which confirms the small percent difference shown previously in Fig. 3 for this case. The radiative flux predicted by the HS approximation is roughly 5% larger than the exact value for this case.

Also shown in Fig. 5 is the spectral dependence of x, which is shown to vary more than 8 orders of magnitude. Note that the majority of the difference between the HS approximation and the exact radiative flux is a result of the 1–2 eV spectral range. In this range, the minimum and maximum x values are roughly 0.001 and 15.0. However, it is mainly the x values between 0.1 and 1.5 that result in the radiative flux inaccuracies for the HS approximation. This is because both the optically thin [Eq. (8)] and optically thick [Eq. (9)] asymptotes are exactly reproduced by the HS approximation. Therefore, it is the inaccuracy in E_3 for these moderate values of x, as shown in Fig. 2, that results in the inaccuracy in the radiative flux prediction.

To confirm that the analysis of the equilibrium constant property layers presented here is relevant to actual nonequilibrium shock layers, the wall-directed radiative flux along the stagnation line of a 1-m-radius sphere in air at 11 km/s and a freestream density of 3×10^{-4} kg/m^3 is compared in Fig. 6. The thermochemical nonequilibrium flowfield was computed using the LAURA Navier–Stokes solver. Radiative cooling and ablation products were not treated [15]. Most of the shock layer is in thermochemical equilibrium for this case, with a temperature and pressure of roughly 11,000 K and 0.3 atm, respectively. The boundary layer, located between $z = 0.0$ and 0.3 cm, represents the primary difference from the constant-property-layer cases. The result obtained with the new approximation is indistinguishable from the exact result, while that of the HS approximation is 5% higher at the wall ($z = 0$). This result is consistent with that observed in the previous discussion of the constant property layers. Note that the result of the HS approximation is shown for this example because it has been the most widely applied in past shock-layer radiation codes [3,4].

IV. Conclusions

A new approximation for the third-order exponential integral, E_3, is proposed for the application of tangent-slab radiation predictions. This new approximation maintains a form that allows for simplifications in the radiation transport equations. For a range of conditions typical of lunar-return shock layers, the new approximation is shown to produce radiative flux values with errors of less than 0.5% compared with the results of the exact E_3 function. This improves upon the 5–20% errors resulting from previous approximations.

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